Discrete Applied Mathematics 37/38 (1992) 287-296
North-Holland

Forwarding indices of k-connected graphs*

M.C. Heydemann, J.C. Meyer

LRI, UA 410 CNRS, bât 490, Université Paris Sud, 91405 Orsay Cedex, France

J. Opatrny

Department of Computer Sciences, Concordia University, Montréal, Canada

D. Sotteau

LRI, UA 410 CNRS, bât 490, Université Paris Sud, 91405 Orsay Cedex, France

Received 21 December 1988
Revised 20 July 1989

Abstract

Heydemann, M.C., J.C. Meyer, J. Opatrny and D. Sotteau, Forwarding indices of k-connected graphs, Discrete Applied Mathematics 37/38 (1992) 287-296.

For a given connected graph G of order n, a routing R is a set of $n(n-1)$ simple paths one specified for each ordered pair of vertices in G; the pair (G, R) is called a network. The vertex (respectively edge) forwarding index $\xi(G, R)$ (respectively $\pi(G, R)$) of a network (G, R) is the maximum number of paths of R passing through any vertex (respectively edge).

In this paper we give upper bounds on these parameters, in terms of the number of vertices and the connectivity of the graph, solving some conjectures given in a previous paper.

1. Introduction

A *routing* R of a connected graph G of order n is a set of $n(n-1)$ elementary paths one specified for each ordered pair u, v of vertices of G. $R(u,v)$ will denote the path from u to v in the routing R of G; note that $R(u,v)$ is not necessarily the same as $R(v,u)$. A network (G,R) is defined as a graph G with a routing R. If each path $R(u,v)$ of R is a shortest path from u to v, we say that we have a *routing of shortest paths* and denote it by R_m.

* The work was supported partially by NSERC of Canada and by PRC Math Info of France.

If some nodes or links fail, it is important to know which paths of the network are destroyed, and quite naturally it seems that a "good" routing should not load any vertex or edge too much, in the sense that not too many paths of the routing should go thrcugh it. It order to measure the load of a vertex, Chung, Coffman, Reiman and Simon introduced in [2] the notion of forwarding index.

The *load of a vertex* v in a given routing R of a graph G, denoted by $\xi(G,R,v)$, is the number of paths of R going through v (where v is not an endvertex). The *vertex forwarding index of a network* (G,R) is the maximum number of paths of R going through any vertex v in G and is denoted by $\xi(G,R)$:

$$\xi(G,R)=\max_{v\in V(G)}\xi(G,R,v).$$

For a graph G, the minimum of the forwarding indices of the networks (G,R), taken over all possible routings R, is denoted by $\xi(G)$ and called the *vertex forwarding index of* G; the minimum taken over all the routings of shortest paths is denoted by $\xi_m(G)$.

$$\xi(G)=\min_R \xi(G,R) \quad \text{and} \quad \xi_m(G)=\min_{R_m}\xi(G,R_m).$$

Similarly, and as in [3], the *load of an edge* e in a given routing R of G, denoted $\pi(G,R,e)$, is the number of paths of R which go through it. The *edge forwarding index* of (G,R), denoted by $\pi(G,R)$, is the maximum number of paths of R going through any edge of G:

$$\pi(G,R)=\max_{e\in E(G)}\pi(G,R,e),$$

and the edge forwarding indices of G are:

$$\pi(G)=\min_R \pi(G,R) \quad \text{and} \quad \pi_m(G)=\min_{R_m}\pi(G,R_m).$$

For definitions and notations not given here, see [1].

If A and B are subsets of vertices of G, $R(A,B)$ will denote the set of paths of R between A and B (from A to B and from B to A). An edge joining two vertices u and v will be denoted uv, and a path will be denoted by the sequence of its vertices, for example, $P=x_0x_1\dots x_{n-1}$. All the graphs considered in this paper are connected if nothing else is specified.

It is easy to observe that if R_1 is a routing of a graph G such that $\xi(G,R_1)=\xi(G)$ ($\pi(G,R_1)=\pi(G)$ respectively), then there exists a routing R_2 such that $\xi(G,R_2)=\xi(G)$ ($\pi(G,R_2)=\pi(G)$ respectively), and

$$R_2(x,y)=R_2(y,x)=xy \quad \text{for each edge } xy \text{ of } G. \tag{1}$$

Thus in the sequel we will only consider routings satisfying (1).

In Sections 2 and 3 of this paper we give bounds on the vertex and edge forwarding indices of a graph G in terms of its number of vertices and its connectivity. Some of these bounds were conjectured in [3].

2. Vertex forwarding index

Let us recall the following result of [3].

Theorem 2.1. *If G is a 2-connected graph of order n, then* $\xi(G) \leq (n-2)(n-3)/2$ *and this bound is best possible in view of* $K_{2,n-2}$.

In the case of routings of shortest paths the following conjecture was stated.

Conjecture 2.2. If G is a 2-connected graph of order $n \geq 6$, then

$$\xi_m(G) \leq n^2 - 7n + 12.$$

This conjecture is proved in [3] for graphs of diameter 2. In this section we prove it in the general case when $n \geq 7$ and we give a sharper bound for graphs of higher connectivity. We need the following lemma.

Lemma 2.3. *Let G be a connected graph of order n. For any two vertices x and y of G and any routing* R_m *of shortest paths in G, we have*

$$\xi(G, R_m, x) + \xi(G, R_m, y) \leq \frac{3n^2 - (10 + 2c)n + c^2 + 8}{2},$$

where c is the number of vertices equidistant from x and y. Moreover if G is 2-connected, then

$$\xi(G, R_m, x) + \xi(G, R_m, y) \leq \frac{3n^2 - (14 + 2c)n + c^2 + 4c + 16}{2}.$$

Proof. Let x and y be two vertices of G and R a routing of shortest paths in G. Let

$$A = \{u \in V(G) - \{x\} \mid d(x, u) < d(y, u)\},$$

$$B = \{u \in V(G) - \{y\} \mid d(x, u) > d(y, u)\},$$

$$C = \{u \in V(G) \mid d(x, u) = d(y, u)\},$$

and $a = |A|$, $b = |B|$, $c = |C|$. We have $A \cup B \cup C = V(G) - \{x, y\}$. Since R is a routing of shortest paths in G, a path of R going through x cannot have both endvertices in B, nor one endvertex in C and the other in B. Let r be the number of paths of R going through x and with both endvertices in C. We have

$$\begin{aligned} \xi(G, R, x) &\leq |R(A, A) \cup R(A, y) \cup R(A, C) \cup R(A, B)| + r \\ &\leq a(a-1) + 2a + 2ac + 2ab + r, \end{aligned}$$

and similarly

$$\xi(G, R, y) \leq b(b-1) + 2b + 2bc + 2ab + r',$$

where r' is the number of paths of R going through y and with both endvertices in C. Since a shortest path in G with both endvertices in C cannot go through both x and y, we have $r+r' \leq c(c-1)$. So

$$\begin{aligned}\xi(G,R,x)+\xi(G,R,y) &\leq a(a-1)+b(b-1)+c(c-1)+4ab+2ac+2bc\\ &\quad +2a+2b\\ &= a^2+b^2+c^2+4ab+2ac+2bc+a+b-c.\end{aligned}$$

Since $n=a+b+c+2$ we have

$$\xi(G,R,x)+\xi(G,R,y) \leq (n-2)^2+2ab+a+b-c$$

and

$$2ab+a+b \leq \frac{(n-2-c)^2}{2}+n-2-c.$$

Therefore

$$\xi(G,R,x)+\xi(G,R,y) \leq \frac{3n^2-(10+2c)n+c^2+8}{2}.$$

If G is 2-connected, $G-x$ is connected. Consider a spanning tree T of $G-x$. The tree T contains at least a edges each of which has at least one end in A. For such an edge uv the paths between u and v do not go through x. Therefore the upper bound of $\xi(G,R,x)$ can be reduced by $2a$; similarly, the upper bound on $\xi(G,R,y)$ can be reduced by $2b$. This observation yields the upper bound

$$\xi(G,R_m,x)+\xi(G,R_m,y) \leq \frac{3n^2-(14+2c)n+c^2+4c+16}{2}. \qquad \square$$

Theorem 2.4. *For any 2-connected graph of order $n \geq 7$,*

$$\xi_m(G) \leq n^2-7n+12$$

and this bound is best possible since it is reached for a fan of order n (that is, the join of a vertex and a path of order $n-1$).

Proof. The proof is by contradiction. Let G be a 2-connected graph of order n such that $\xi_m(G) \geq n^2-7n+13$. Let R be a routing of shortest paths of G such that $\xi(G,R)=\xi_m(G)$ in which the minimum number of vertices z of G satisfy $\xi(G,R,z)=\xi(G,R)$. Let x be a vertex of G such that $\xi(G,R,x)=\xi(G,R)$. As in the proof of [3, Theorem 4.2], the number of couples of vertices at distance more than 2 in $G-x$ is at most $n^2-7n+12$. Therefore there exists a couple (u,v) of vertices of $G-x$ such that $R(u,v)=uxv$ and a vertex y of $G-x$ adjacent to both u and v. By Lemma 2.3, since x and y have two common neighbours u and v,

$$\xi(G,R,y) \leq \frac{3n^2-18n+28}{2}-\xi(G,R,x) \leq \frac{n^2}{2}-2n+1.$$

But, for $n \geq 7$, $n^2/2 - 2n + 1 < n^2 - 7n + 12$. So the routing R' which is derived from R only by replacing $R(u, v)$ by $R'(u, v) = uyv$, is a routing of shortest paths such that $\xi(G, R') = \xi_m(G)$ for which fewer vertices z of G satisfy $\xi(G, R', z) = \xi_m(G)$. This contradicts the hypothesis on R. □

We now give some results for k-connected graphs.

Lemma 2.5. *Let G be a k-connected graph of order n. The number of couples of distinct vertices at distance at most* 2 *in G is at least* $\min(2kn, n(n-1))$.

Proof. For every x in $V(G)$, let $\sigma(x) = |\{y \in V(G) \mid y \neq x,\ d(x, y) \leq 2\}|$. If $\sigma(x) < n - 1$ there exists a vertex y in G such that $d(x, y) \geq 3$. By Menger's theorem there exist at least k vertex disjoint paths with endvertices x and y. On each path there are at least two distinct vertices at distance 1 or 2 from x. Therefore $\sigma(x) \geq 2k$. So, for every x in $V(G)$, we have $\sigma(x) \geq \min(2k, n-1)$ and the number of couples of vertices at distance at most 2 in G, namely $\sum_{x \in V(G)} \sigma(x)$, is at least $\min(2kn, n(n-1))$. □

Theorem 2.6. *For any k-connected graph G of order n with* $k \geq 3$ *and* $n \geq 8k - 10$, $\xi_m(G) \leq n^2 - (2k+1)n + 2k$.

The proof is exactly the same as that of Theorem 2.4, using Lemmas 2.3 and 2.5. We note that this bound is best possible for all odd k, as can be seen from the $(2p+1)$-connected graph formed by joining one vertex to all vertices of C_{n-1}^p, the pth power of a cycle of length $n-1$; the graph C_{n-1}^p is $2p$-connected (see for example [4]).

The best upper bound on $\xi(G)$ for 2-connected graphs is $(n-2)(n-3)/2$ which is much smaller than the best upper bound on $\xi_m(G)$ namely $n^2 - 7n + 12$, given in Theorem 2.4. Similarly we can state the following problem for k-connected graphs.

Problem 2.7. Find the best upper bound $f(n, k)$ such that for any k-connected graph G of order n with $k \geq 3$, $\xi(G) \leq f(n, k)$ for n large enough compared to k.

Manoussakis and de la Vega [6] have proved that $f(n, k) \leq (n-1)(n-k-1)/k$ and they conjecture the bound $f(n, k) \leq (n-k)(n-k-1)/k$ which would be best possible in view of the complete bipartite graph $K_{k, n-k}$.

3. Edge forwarding index

We will give now a solution of a conjecture from [3] on the best upper bound on $\pi(G)$ for 2-connected graphs.

Theorem 3.1. *If G is a 2-edge connected graph of order n, then* $\pi(G) \leq \lfloor n^2/4 \rfloor$ *and this bound is best possible in view of the cycle* C_n *(see* [3]).

We will first give four lemmas which will be needed for the proof of Theorem 3.1. The following result is well known [5].

Lemma 3.2. *If G is a 2-connected graph which is not a cycle, then $G=G_1\cup G_2\cup\cdots\cup G_r$ where G_1 is a cycle and G_i, $2\le i\le r$, is a path having exactly its endvertices in common with $G_1\cup G_2\cup\cdots\cup G_{i-1}$. Furthermore each cycle of G can be chosen as cycle G_1.*

Lemma 3.3. *Let G be an edge critically 2-connected graph of order $n\ge 5$ which is not a cycle. Then G contains a path P and a cycle C such that*

(i) *the internal vertices of P are of degree* 2,

(ii) *the endvertices of P are on C,*

(iii) *if t is the length of P, then $2\le t\le (n+1)/3$.*

Proof. By Lemma 3.2, $G=G_1\cup G_2\cup\cdots\cup G_r$ where G_1 is a longest cycle of G and G_{i+1}, $i\ge 1$, is a path having exactly its endvertices in common with $G_1\cup G_2\cup\cdots\cup G_i$. Since G is edge critically 2-connected, the length of G_{i+1}, $i\ge 1$, is greater than 1. The length of G_r is less than $(n+1)/3$, otherwise we could find a cycle of G containing G_r with more vertices than G_1 contradicting the choice of G_1. $\square$

Lemma 3.4. *Let G be a graph of order $n\ge 5$ which contains a path P and a cycle C satisfying the conditions* (i), (ii) *and* (iii) *of Lemma* 3.3. *Let G' be the graph obtained from G as follows. If $P=p_0p_1p_2\ldots p_t$ with p_0 and p_t on C and if a and b are respective neighbours of p_0 and p_t on the same subpath of C delimited by p_0 and p_t, G' is the graph obtained from G by deleting the vertices $p_1,p_2,\ldots,p_{t-1}$ and replacing the edges p_0a and p_tb of C respectively by paths P_1a with $P_1=p_0p_1p_2\ldots p_m$ and bP_2 with $P_2=p_{m+1}\ldots p_{t-1}p_t$ where $m=\lfloor t/2\rfloor$. Then $\pi(G)\le\max(\pi(G'),\ \lfloor n^2/4\rfloor)$.*

Proof. Let R' be a routing of G' such that $\pi(G',R')=\pi(G')$. We will define a routing R of G such that

- $\forall e\in G-(P\cup\{p_0a,p_tb\})$, $\pi(G,R,e)\le\pi(G',R',e)$,
- $\forall e\in P$, $\pi(G,R,e)\le\lfloor n^2/4\rfloor$,
- $\pi(G,R,p_0a)\le\pi(G',R',p_ma)$,
- $\pi(G,R,p_tb)\le\pi(G',R',p_{m+1}b)$.

The lemma will then follow immediately.

We define a routing R of G as follows. Consider any two vertices x and y of $G-P$. If in G', $R'(x,y)$ does not go through any edge of P_1 or P_2, then we take $R(x,y)=R'(x,y)$. Note that if $R'(x,y)$ contains one edge of P_1, it necessarily contains P_1a since all interval vertices of P_1a are of degree 2 (and similarly for P_2). If $R'(x,y)$ contains P_1a (respectively P_2b), then we take for $R(x,y)$ in G the path obtained from $R'(x,y)$ by replacing P_1a (respectively P_2b) by the edge p_0a (respectively p_tb).

We now define the paths of R between any vertex p_i of P and any vertex x of $G-P$. If p_i is in P_1 and we have $R'(p_i,x)=p_ip_{i+1}\dots p_m aQ$ in G', then we take $R(p_i,x)=p_ip_{i-1}\dots p_0 a\bar{Q}$ in G, where $\bar{Q}$ is in G the same path as Q, with possibly the subpath P_2b (or bP_2) replaced by the edge p_tb (or bp_t). If $R'(p_i,x)=p_ip_{i-1}\dots p_0Q$ in G', then we take $R(p_i,x)=p_ip_{i-1}\dots p_0\bar{Q}$ in G. If p_i is in P_2, we take $R(p_i,x)=p_ip_{i+1}\dots p_tb\bar{Q}$ if $R'(p_i,x)=p_ip_{i-1}\dots p_{m+1}bQ$ and $R(p_i,x)=p_ip_{i+1}\dots p_t\bar{Q}$ if $R'(p_i,x)=p_ip_{i+1}\dots p_tQ$, where $\bar{Q}$ is the same path as Q, with possibly the subpath P_1a (or aP_1) replaced by the edge p_0a (or ap_0). We perform a similar construction to obtain the path $R(x,p_i)$.

Finally, let us define the paths of R between two vertices u and v of P to be the subpaths $P(u,v)$ and $P(v,u)$ of P.

Clearly, the load induced by all these paths of R on edges of $G-(P\cup\{p_0a,p_tb\})$ is at most equal to the one induced by the corresponding paths of R' in G' and the load induced on p_0a (respectively p_tb) is at most the one induced in G' on p_ma (respectively $p_{m+1}b$).

Let us show now that, for any edge $e_i=p_ip_{i+1}$, $0\le i\le t-1$, of P, $\pi(G,R,e_i)\le \lfloor n^2/4\rfloor$.

For any i, $1\le i\le m-1$ we have

$$\pi(G,R,e_i)\le \pi(G,R,e_{i-1})-2(n-(t+1))+2(t-2i).$$

Indeed the load induced on e_i by paths between vertices of P is $2(i+1)(t-i)$ and the one induced on e_{i-1} is $2i(t-i+1)$. All the paths between a vertex of P and a vertex of $G-P$ which go through e_i also go through e_{i-1}; moreover, the paths between p_i and $G-P$ go through e_{i-1} but not through e_i. So we have $\pi(G,R,e_i)\le \pi(G,R,e_{i-1})-2n+4t-4i+2\le \pi(G,R,e_{i-1})$ since $t\le\lfloor (n+1)/3\rfloor$.

Similarly, for any i, $\lceil t/2\rceil\le i\le t-2$, we have $\pi(G,R,e_i)\le\pi(G,R,e_{i+1})$. Indeed, we have

$$\begin{aligned}\pi(G,R,e_i)&\le \pi(G,R,e_{i+1})-2(n-t-1)+2(i+1)(t-i)-2(i+2)(t-i-1)\\&\le \pi(G,R,e_{i+1})-2n+4i+6\\&\le \pi(G,R,e_{i+1})-2n+4t-2\\&\le \pi(G,R,e_{i+1})\end{aligned}$$

since $t\le(n+1)/3$. Thus e_0 and e_{t-1} are the most loaded edges of P. We have

$$\pi(G,R,e_0)=2t+2\left\lfloor\frac{t}{2}\right\rfloor(n-t-1)\le t(n-t+1)$$

and, therefore, $\pi(G,R,e_0)\le n^2/4$ for any $2\le t\le(n+1)/3$ when $n\ge 9$ and, similarly, for $\pi(G,R,e_{t-1})$. For $5\le n\le 8$ it is possible to modify the routing only between vertices of P to get the same result. □

Lemma 3.5. *For any 2-edge connected graph G there exists a 2-connected graph G' such that $\pi(G)\le\pi(G')$.*

Proof. Let a be a cut vertex of G and G_1 and G_2 two connected components of $G-a$. Let b and c be two vertices of G_1 and G_2 adjacent to a. Remark that since G is 2-edge connected there exist other edges than ab and ac between a and G_1 and G_2. Let G' be the graph obtained from G by deleting the edges ab and ac and adding the edge bc. Let R' be a routing of G' such that $\pi(G',R')=\pi(G')$. Define a routing R of G from R' by replacing the edge bc in any route of R' containing it by the path bac. This routing satisfies $\pi(G,R)\leq\pi(G',R')=\pi(G')$, but the paths between a and b, a and c respectively, are not the edge ab, ac respectively. However as observed in the first section this routing can be modified to give another one satisfying equality (1) of Section 1. This shows that $\pi(G)\leq\pi(G')$.

Therefore $\pi(G)\leq\pi(G')$. If G' is not 2-connected we can repeat the above construction until we obtain a 2-connected graph. □

Proof of Theorem 3.1. By Lemma 3.5 we assume that G is 2-connected. Let G be a 2-connected graph of order n. We delete edges of G until we get a subgraph H of G which is edge critically 2-connected. Obviously, $\pi(G)\leq\pi(H)$.

Now, in the case $n\geq 5$, we construct a sequence of edge critically 2-connected graphs of order n, $H_0=H, H_1, H_2$ and so on, where, at each step, H_i is obtained from H_{i-1} in the same way as G' is obtained from G in Lemma 3.4. This sequence is clearly finite and will end with a cycle C of length n. By applying Lemma 3.4 at each step we have $\pi(H)\leq\max(\pi(H_1),\lfloor n^2/4\rfloor)\leq\max(\pi(H_2),\lfloor n^2/4\rfloor)\leq\cdots\leq\max(\pi(C),\lfloor n^2/4\rfloor)$. Since $\pi(C)=\lfloor n^2/4\rfloor$ we have proved the theorem for $n\geq 5$.

In the case $n\leq 4$ the edge critically 2-connected graphs are themselves cycles so the result is clear. □

If we only consider routings of shortest paths, we have the following result [3].

Theorem 3.6. *If G is a 2-edge connected graph, then*

$$\pi_m(G)\leq\left\lfloor\frac{n^2}{2}-n+\frac{1}{2}\right\rfloor$$

and this bound is best possible in view of the following graph G_0. G_0 is the disjoint union of two complete graphs on respectively $\lfloor(n-1)/2\rfloor$ and $\lceil(n-1)/2\rceil$ vertices and an extra vertex c, with an edge ab between the complete graphs and two edges ac and bc.

In the case of 2-connected graphs the upper bound on $\pi_m(G)$ cannot be improved substantially, as shown by an example in [3]. We give another example of a 2-connected graph G_1 of order n for which $\pi_m(G_1)=\lfloor n^2/2-n-11/2\rfloor$. G_1 is the disjoint union of two complete graphs of order $\lfloor(n-3)/2\rfloor$ and $\lceil(n-3)/2\rceil$ with three extra vertices a, b and c. If u, v and u', v' are respectively distinct vertices of the two complete graphs, G_1 contains a path u,a,b,c,u' and edges uu', va and cv'.

In the case of k-connected graphs we state the following problem.

Problem 3.7. Find the best functions $g(n,k)$ and $h(n,k)$ such that for any k-connected graph G of order n with $k \geq 2$, $\pi(G) \leq g(n,k)$ and $\pi_m(G) \leq h(n,k)$ for n large enough compared to k.

The same problem can be considered for k-edge connected graphs.
Let us recall the following conjecture [3].

Conjecture 3.8. For any k-edge connected graph G of order n,

$$\pi_m(G) \leq \left\lfloor \frac{n^2}{2} - (k-1)n + \frac{(k-1)^2}{2} \right\rfloor .$$

This conjecture is false. Indeed the edge forwarding index of each of the two k-edge connected graphs G_0 and G_1 that we will exhibit now is greater than the conjectured upper bound.

Let G_0 be the graph containing four complete graphs A, B, C, D and a vertex x with $|A| = \lfloor (n-2k+1)/2 \rfloor$, $|B| = \lceil (n-2k+1)/2 \rceil$, $|C| = |D| = k-1$. Let a be a vertex of A, b be a vertex of B. Vertices of C are connected to a and x, vertices of D are connected to b and x, and furthermore, G_0 contains the edge ab. It is not difficult to verify that G_0 is k-edge connected for $n \geq 4k+1$ and that

$$\pi_m(G_0) = \left\lceil \frac{n^2}{2} \right\rceil - n - 2k^2 + 4k - 2.$$

Let G_1 be the graph containing five complete graphs A, B, C, D and K_2 with $|A| = \lfloor (n-k-1-\varepsilon)/2 \rfloor$, $|B| = \lceil (n-k-1-\varepsilon)/2 \rceil$, $|C| = |D| = \lceil (k-1)/2 \rceil$, where $\varepsilon = 1$ if k is even and $\varepsilon = 0$ otherwise. Let A' be a set of $\lfloor (k-1)/2 \rfloor$ vertices of A, B' a set of $\lfloor (k-1)/2 \rfloor$ vertices of B, a a vertex of A' and b a vertex of B'. G_1 contains the edge ab and all the edges between $K_2 \cup A'$ and C and between $K_2 \cup B'$ and D. It is not difficult to verify that G_1 is k-edge connected for $n \geq 3k+3$ and that

$$\pi_m(G_1) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 2n + 5k - \frac{3}{2}(k^2+1)$$

with equality when k is odd.

Since we do not know of any other graph with a greater edge forwarding index than G_0 and G_1, Conjecture 3.8 may be replaced by the following one.

Conjecture 3.9. For any k-edge connected graph G of order n, with $k \geq 3$ and $n \geq 3k+3$,

$$\pi_m(G) \leq \max\left(\left\lceil \frac{n^2}{2} \right\rceil - n - 2k^2 + 4k - 2, \left\lfloor \frac{n^2}{2} \right\rfloor - 2n + 5k - \frac{3}{2}(k^2+1) \right).$$

Note added in proof

Problem 3.7 and Conjecture 3.9 have been partially solved in [6].

Acknowledgement

We thank very much the referees for their useful comments and also A. Bondy for all his suggestions to improve the paper and in particular for the idea of the proof of Lemma 3.5.

References

[1] C. Berge, Graphs and Hypergraphs (Dunod, Paris, 1973).
[2] F.K. Chung, E.G. Coffman, M.I. Reiman and B.E. Simon, The forwarding index of communication networks, IEEE Trans. Inform. Theory 33 (1987) 224.232.
[3] M.C. Heydemann, J.C. Meyer and D. Sotteau, On forwarding indices of networks, Discrete Appl. Math. 23 (1989) 103.123.
[4] J.L. Jolivet, Sur les puissances de graphes connexes, C.R. Acad. Sci. Paris 272 (1971) 107.109.
[5] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).
[6] W. Fernandez de la Vega and Y. Manoussakis, The forwarding index of communication networks with given connectivity, Res. Rept. LRI # 499; also: Discrete Appl. Math. 37/38 (1992) 147-155.